Pergamon
www.elsevier.com/locate/jappmathmech

PII: S0021-8928(01)00077-6

J. Appl. Maths Mechs, Vol. 65, No. 5, pp. 715–721, 2001

0021-8928/01/$—see front matter

GYROSCOPIC STABILIZATION AND PARAMETRIC RESONANCE†

V. V. KOZLOV

Moscow

e-mail: vako@mech.mat.msu.su

(*Received* 19 *December* 2000)

The problem of the conditions for the gyroscopic stabilization of unstable equilibria using gyroscopic forces with a degenerate matrix is considered. Systems with an odd number of degrees of freedom are an important example. The gyroscopic forces can generally be removed using a non-autonomous orthogonal transformation. The equations of motion then become a system of Sturm–Liouville type equations with a time-dependent potential. The conditions imposed on the skew-symmetric matrix of the gyroscopic forces for which the new potential depends periodically on time are indicated. These conditions are necessarily satisfied for non-zero matrices of the gyroscopic forces of minimum rank equal to two. Hence, the problem of gyroscopic stabilization reduces, in a number of cases, to investigating the stability of the equilibrium positions of systems with a periodic potential. The use of parametric-resonance theory enables new constructive conditions to be obtained for the stability of the equilibria of mechanical systems acted upon by additional degenerate gyroscopic forces. These conditions have the form of the conditions for an extremum of certain functions which depend solely on the position of the system. Particular attention is devoted to the stability conditions for large gyroscopic forces. It is shown, using examples, that the conditions of gyroscopic stabilization obtained are only sufficient. However, if the potential energy in the equilibrium position has a maximum and the matrix of the gyroscopic forces are non-degenerate, they are close to the necessary stability conditions.

1. REDUCTION TO THE NON-AUTONOMOUS CASE

As we know [1], small oscillations of a dynamical system in the region of an equilibrium position satisfy the linear equation

$$\ddot{x} + \Gamma\dot{x} + Px = 0, \quad x \in \mathbb{R}^n \tag{1.1}$$

where Γ is a skew-symmetric $n \times n$ matric, while the matrix P is symmetric. The term $-\Gamma\dot{x}$ has the meaning of a gyroscopic force acting on the system.

The point $x = 0$ is an equilibrium position. A review of the results obtained on the problem of the stability of equilibrium can be found in [2]. The case when the potential energy $V = (Px, x)/2$ has a maximum is usually considered.

It turns out that we can generally get rid of the gyroscopic forces if we make the replacement of variables

$$x = A(t)z, \quad A = \exp(-\Gamma t/2) \tag{1.2}$$

In the new coordinates z, Eq. (1.1) takes the form

$$\ddot{z} + Q(t)z = 0, \quad Q = A^{-1}(P - \Gamma^2/4)A \tag{1.3}$$

However, after this replacement, Lagrange's function

$$L = (\dot{z}, \dot{z})/2 - (Qz, z)/2 \tag{1.4}$$

depends explicitly on time. Note that after making the inverse change (1.2) the Lagrangian (1.4) becomes the function

$$L = (\dot{x}, \dot{x})/2 + (\dot{x}, \Gamma x)/2 - (Px, x)/2$$

Lagrange's equation with this Lagrangian is obviously identical with (1.1).

†*Prikl. Mat. Mekh.* Vol. 65, No. 5, pp. 739–745, 2001.

Since the matrix Γ is skew-symmetric, the matrices A and A^{-1} are orthogonal. Consequently, problems of the stability of trivial solutions of Eqs (1.1) and (1.3) are equivalent.

It would seem at first glance that a reduction to a non-autonomous system only complicates the stability analysis. However, the examples given below show that this is not so.

1. We will assume that the matrices Γ and P commute: $\Gamma P = P\Gamma$. Then the matrix Q does not depend explicitly on time and the criterion of gyroscopic stabilization reduces to the following: the modified potential energy

$$W(z) = (Pz, z) + (\Gamma z, \Gamma z)/4 \tag{1.5}$$

has a strict minimum at the point $z = 0$. This result was obtained earlier by another method in [3].

2. Suppose the quadratic form (1.5) is non-positive for all $z \in \mathbb{R}^n$. Then $L \geqslant 0$ and, by a result obtained previously in [4], the equilibrium $z = 0$ of system (1.3) is unstable. Consequently, the equilibrium $x = 0$ of initial system (1.1) is also unstable. This result was obtained in [5] for a stronger condition: the form of (1.5) is negative-definite.

3. We will assume that the matrix $Q(t)$ is periodic with respect to time. In this case the classical Thomson result on the impossibility of gyroscopic stabilization of the equilibrium $x = 0$ for an odd degree of instability can be derived from Hill's formula, which relates the multiplier of the zeroth periodic solution of system (1.3) with its Morse index (see [6, 7]). In connection with this observation, it would be useful to extend Hill's formula to the more general case when the elements of the matrix Q depend conditionally-periodically on time.

2. THE STRUCTURE OF THE GYROSCOPIC FORCES FOR SYSTEMS WITH A PERIODIC POTENTIAL

We will assume

$$\Gamma = \gamma S^T I_k S \tag{2.1}$$

$$I_k = \mathrm{diag}(J, \ldots, J, 0), \quad J = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \quad \mathrm{rank}\, I_k = 2k$$

where γ is a real number and S is an orthogonal $n \times n$ matrix. It can be assumed that $\gamma > 0$: the case $\gamma < 0$ reduces to this after making the replacement $t \to -t$.

Since

$$(S^T I S)^T = -(S^T I S)$$

matrix (2.1) is, in fact, skew-symmetric. We will discuss its properties.

1. The matrix $\exp(\pm\, \Gamma t/2) - (4\pi/\gamma)$ is periodic in t.

In fact, this matrix satisfies the equation

$$\dot{A} = \pm \Gamma A/2, \quad A(0) = E.$$

We will put $A = [u_1, \ldots, u_n]$, where $u_s \in \mathbb{R}^n$ are solutions of the linear system

$$\dot{u} = \pm \Gamma u/2$$

Consequently

$$\dot{u} = \pm \gamma S^T I S u/2$$

We will assume $v = Su$ and use the orthogonality condition $S^T = S^{-1}$.

$$\dot{v} = \pm \gamma I_k v/2$$

If $v = (v_1, \ldots, v_n)^T$, then

$$\dot{v}_1 = \pm \gamma v_2/2, \quad \dot{v}_2 = \mp \gamma v_1/2, \ldots, \dot{v}_{2k-1} = \pm \gamma v_{2k}/2$$

$$\dot{v}_{2k} = \mp \gamma v_{2k-1}/2, \quad \dot{v}_{2k+1} = \ldots = \dot{v}_n = 0$$

The solutions of this system are periodic in t with period $4\pi/\gamma$.

2. If Γ has the form (2.1), the matrix $Q(t)$ from (1.3) is $(2\pi/\gamma)$-periodic in t.

Here we use the $(4\pi/\gamma)$-periodicity of the orthogonal matrices $A(t)$ and $A^{-1}(t)$, and also the fact that the functions $\sin^2$, sin cos and $\cos^2$ have period π.

3. The following inequality hold for all $x \in \mathbb{R}^n$

$$(\Gamma x, \Gamma x) - \gamma^2(x, x) \leqslant 0 \tag{2.2}$$

If fact

$$(S^T ISx, S^T ISx) - (Sx, Sx) = (Iz, Iz) - (z, z) = -z_{2k+1}^2 - \ldots - z_n^2 \leqslant 0$$

Example. We will show that, when $n = 3$, any skew-symmetric matrix Γ has the form (2.1), where $k = 1$, if $\Gamma \neq 0$.

Obviously, it is sufficient to consider the case when

$$\Gamma = \begin{Vmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{Vmatrix} \tag{2.3}$$

and $\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 = 1$. In (2.1) suppose $\gamma = 1$ and

$$S = \begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{Vmatrix}$$

Then $\omega = b \times a$, when ω, a and b are vectors of a three-dimensional Euclidean space with components ω_i, a_i and b_i respectively.

In view of the orthogonality of the matrix S

$$|a| = |b| = 1 \ \text{ and } \ (a, b) = 0 \tag{2.4}$$

It is clear that for any unit vector ω we can always find two vectors a and b which satisfy conditions (2.4) such that $\omega = b \times a$.

This example can be generalized. We will show that if $\operatorname{rank}\Gamma = 2$, the matrix Γ can be represented in the form (2.1).

The vector ω is called the vortex vector if $\Gamma\omega = 0$. All vortex vectors form a linear space W of dimension $n - 2$. Suppose the vectors $u = (u_1, \ldots, u_n)^T, \ldots, v = (v_1, \ldots, v_n)^T$ make up an orthonormalized basis in W. We will supplement its vectors $a = (a_1, \ldots, a_n)^T$ and $b = (b_1, \ldots, b_n)^T$ up to the orthonormalized basis in $\mathbb{R}^n$. The matrix

$$\begin{Vmatrix} a_1 & b_1 & u_1 & \ldots & v_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & b_n & u_n & \ldots & v_n \end{Vmatrix}$$

is obviously orthogonal. Denote it by S^{-1}. The product ΓS^{-1} has the form

$$\| * \ \ * \ \ 0 \|$$

where the asterisks denote two non-zero columns. The skew-symmetric matrix $S(\Gamma S^{-1})$ has the same form. Consequently, $S\Gamma S^T = \gamma I_2$, whence Eq. (2.1) follows.

We will now show that the problem of the gyroscopic stabilization of an unstable equilibrium is closely related to the phenomenon of parametric resonance. To do this we will consider the case when $\det\Gamma \neq 0$ (consequently, n is even), and the elements of the matrix P are small. The latter is equivalent to the assumption that the potential energy takes moderate values, while the norm of the matrix of the gyroscopic forces is large. Taking (2.1) and property 2 into account, Eq. (1.3) can be reduced to the form

$$\ddot{y} + \left(\frac{\gamma^2}{4} E + R(\gamma t)\right) y = 0, \quad y \in \mathbb{R}^n \tag{2.5}$$

where $y = Sz$ and $R(\omega)$ is a 2π-periodic symmetric $n \times n$ matrix. By our assumption, the norm $\|P\|$ is small; consequently, the norm $\|R\|$ is also small.

Equation (2.5) describes the oscillations of a mechanical system with a natural frequency $\Omega = \gamma/2$ acted upon by a small periodic perturbation with frequency $\omega = \gamma$. When $\|P\| \to 0$ we have parametric resonance $2\Omega = \omega$. Moreover, the system is at the boundary of the most dangerous zone of parametric resonance, when the frequency of the natural oscillations is half the frequency of the perturbing force. Hence, the possibility of gyroscopic stabilization of the equilibrium $x = 0$ will depend on where system (2.5) is when the elements of the matrix P increase: outside or inside the zone of parametric resonance.

3. THE STABILITY CONDITIONS

The main result is as follows.

Theorem. Suppose the matrix of the gyroscopic forces has the form (2.1). If $x = 0$ is a strict minimum of the modified potential energy $W(x)$ and a strict maximum of the difference

$$W(x) - \gamma^2(x, x)/4 \tag{3.1}$$

then $x = 0$ is a stable equilibrium of system (1.1).

In order to understand better the meaning of the sufficient conditions of stability, we will consider the limiting case when the parameter γ takes larger values. We will introduce the plane

$$\Lambda = \{x: \Gamma x = 0\}$$

Suppose M is the orthogonal supplement of Λ. It can be shown that

$$M = \{x : x = \Gamma z, z \in \mathbb{R}^n\}$$

In fact

$$(x, \Gamma z) = -(\Gamma x, z) = 0$$

if $x \in \Lambda$. Moreover

$$\dim M = \operatorname{rank}\Gamma = 2k, \quad \dim\Lambda = n - \operatorname{rank}\Gamma$$

For large values of γ the condition for the modified potential energy at the point $x = 0$ to be a minimum becomes the condition

$$V(x) > 0 \quad \text{for all} \quad x \in \Lambda \quad \text{and} \quad x \neq 0 \tag{3.2}$$

By virtue of inequality (2.2), the function (3.1) has a maximum for large values of γ, if

$$V(x) < 0 \quad \text{for all} \quad x \in M \quad \text{and} \quad x \neq 0 \tag{3.3}$$

This condition can be represented in the following equivalent form: the quadratic form $(P\Gamma z, \Gamma z)$ is negative-definite in the subspace M.

Hence, according to the theorem, if conditions (3.2) and (3.3) are satisfied, then for large values of Γ the equilibrium $x = 0$ is stable. Somewhat different sufficient conditions for stability for large gyroscopic forces were obtained previously in [8] by the method of Lyapunov functions.

Note that conditions (3.2) and (3.3) are only the sufficient conditions for stability.

In fact, suppose $n = 3$, $P = \operatorname{diag}(p_1, p_2, p_3)$, while the matrix Γ has the form (2.3). As was shown in [8], if the degree of instability is even, the criterion of gyroscopic stabilization of the equilibrium $x = 0$ in the case of large values of Γ reduces to the inequality

$$p_1\omega_1^2 + p_2\omega_2^2 + p_3\omega_3^2 > 0$$

It can be shown that this inequality is equivalent to condition (3.2).

It is worth mentioning that it follows from conditions (3.2) and (3.3) that the degree of Poincaré instability for the potential V is even. Consequently, the necessary condition for Kelvin gyroscopic stabilization is necessarily satisfied.

We will now prove the theorem. By the condition, all the eigenvalues of the symmetrical matrix $P-\gamma^2/4$ are positive. We will denote them ω_s^2 ($\omega_s > 0$) and arrange them in increasing order

$$0<\omega_1^2 \leqslant \omega_2^2 \leqslant \ldots \leqslant \omega_n^2 \tag{3.4}$$

Since the matrix $A(t)$ is orthogonal, then (according to (1.3)) the inequalities

$$\omega_1^2(x,x) \leqslant (Q(t)x,x) \leqslant \omega_n^2(x,x) \tag{3.5}$$

hold for all values of t.

By property 2 (Section 2) the frequency of the periodic matrix $Q(t)$ is

$$\omega = 2\pi/T = \gamma$$

Since the quadratic form (3.1) is negative-definite and ω_n^2 is the maximum eigenvalue of the matrix $P-\Gamma^2/4$, we have

$$\omega > 2\omega_n \tag{3.6}$$

It remains to use the well-known result that, when conditions (3.5) and (3.6) are satisfied, the trivial solution $z = 0$ of system (1.3) is stable.

Remark. The sufficiency of conditions (3.5) and (3.6) for the stability of the trivial solution of a system of the form (1.3) can be simply derived from the general results of the strong stability of linear Hamiltonian systems with periodic coefficients (see [9, Chapter III]). They are given in explicit form, for example, in [10].

As an example, consider the case when the potential energy in equilibrium has a strict maximum and the matrix of the gyroscopic forces (2.1) is degenerate. In particular, n is even. In view of inequality (2.2) and the assumption that the potential V is negative-definite, the quadratic form (3.1) is negative-definite. Consequently, in this case the sufficient condition for gyroscopic stabilization is the condition that the changed potential energy should be positive-definite. However, it is now the necessary condition.

Example. Suppose $n = 2$, $P = \mathrm{diag}(p_1, p_2)$ and the matrix Γ always has the form (2.1):

$$\Gamma = \begin{Vmatrix} 0 & \gamma \\ -\gamma & 0 \end{Vmatrix}$$

Here the orthogonal matrix S can be assumed to be unique. In the most interesting case, the potential energy has a maximum: the numbers p_1 and p_2 are negative. As we know (see, for example, [1]), the stability condition reduces to the inequality

$$\gamma > \sqrt{|p_1|} + \sqrt{|p_2|} \tag{3.7}$$

On the other hand, the condition for the matrix $P-\Gamma^2/4$ to be positive-definite is equivalent to the inequality

$$\gamma > 2\sqrt{|p_1|}, \quad p = \min(p_1, p_2) \tag{3.8}$$

This is obviously stronger than condition (3.7) and is only identical with it when $p_1 = p_2$.

4. SOME GENERALIZATIONS

It was noted in [10] that conditions (3.5) and (3.6) for the stability of the zeroth solution of system (1.3) can be weakened somewhat: the inequality on the left in (3.5) can be replaced by the condition for the averaged matrix

$$\langle Q \rangle = \frac{1}{T}\int_0^T Q(t)dt, \quad T = \frac{2\pi}{\gamma} \tag{4.1}$$

to be positive-definite. Hence, in the theorem in Section 3 the condition for the quadratic form W to be positive-definite can be replaced by the weaker condition for the symmetric matrix $\langle Q \rangle$ to be positive.

Example. Consider once again the case when $n = 2$ with the assumptions and notation of Section 3. It can be shown that

$$\langle Q \rangle = \frac{p_1 + p_2}{2} E$$

where E is the 2×2 identity matrix.

Hence, we obtain the sufficient condition for gyroscopic stabilization

$$\gamma > \sqrt{2 | p_1 + p_2 |} \tag{4.2}$$

When $p_1 \neq p_2$ we have the inequalities

$$2\sqrt{| p |} > \sqrt{2 | p_1 + p_2 |} > \sqrt{| p_1 |} + \sqrt{| p_2 |}$$

Consequently, condition (4.2) is more accurate than conditions (3.3), but it is also not necessary for the stabilization of the equilibrium.

The theorem of Section 3 can be refined further. Suppose ω_1 and ω_n are the minimum and maximum eigenvalues of the positive-definite matrix (3.4), respectively. Inequalities (3.5) enable us to use well-known stability conditions for parametric excitation [9, 10]: if the frequency of the periodic excitation $\omega = \gamma$ does not lie in any of the intervals

$$2[\omega_1, \omega_n]/m, \quad m = 1, 2, \ldots \tag{4.3}$$

the trivial solution $z = 0$ is stable. Consequently, this condition is sufficient for the rest point $x = 0$ of the initial system to be stable. In particular, if condition (3.6) is satisfied, the frequency γ must necessarily lie at the union of intervals (4.3).

In view of the fact that the harmonic series diverges, the intervals (4.3) necessarily overlap. However, the measure of their union approaches zero when $\omega_n - \omega_1 \to 0$.

5. THE CONDITIONS OF GYROSCOPIC STABILIZATION, BASED ON ESTIMATES OF THE EIGENVALUES

To investigate the stability of the trivial solution of Eq. (1.3) with T-periodic symmetric matrix $Q(t)$, we can use Krein's criteria [11] (see also [9, Chapter III]). From the point of view of the form of the matrix Q, criterion 4 from [11], which, incidentally, is inaccurately formulated in [11] and [9], is the most constructive. Here is its correct formulation: if the least eigenvalue $q(t)$ of the matrix $Q(t)$ satisfies the conditions

$$q(t) \geqslant a^2 > 0 \tag{5.1}$$

and

$$0 < a < \pi/T \tag{5.2}$$

linear equation (1.3) is extremely stable, since

$$\frac{1}{T}\int_0^T \operatorname{tr} Q(t)dt < \frac{4}{T^2} + \left(n - \frac{4}{\pi^2}\right)a^2 \tag{5.3}$$

We will use this criterion for matrix Q of the form (1.3). Clearly, for all values of t the spectrum of the matrix $Q(t)$ is identical with the spectrum of the constant matrix (3.4). Consequently, condition (5.1) is satisfied if the modified potential energy $W(x)$ has a strict minimum at the point $x = 0$. In the notation of Section 3, $a = \omega_1$.

We will show that condition (5.2) is necessarily satisfied if the potential energy $V(x)$ does not have a minimum at the point $x = 0$. It is precisely this case that is of interest in the problem of gyroscopic stabilization. In fact, suppose

$$a \geqslant \pi/T = \gamma/2$$

Then, by condition (5.1), we have the inequality

$$4(Px, x) + (\Gamma x, \Gamma x) \geqslant \gamma^2(x, x)$$

Consequently (using (2.2)), $V(x) \geqslant 0$. We have obtained a contradiction.

Since

$$\operatorname{tr}Q(t) = \operatorname{tr}(P - \Gamma^2/4) = \text{const}$$

inequality (5.3) gives the following sufficient condition for gyroscopic stabilization

$$\operatorname{tr} P + \frac{k\gamma^2}{2} < \frac{\gamma^2}{\pi^2} + \left(n - \frac{4}{\pi^2}\right)\omega_1^2 \tag{5.4}$$

Here $k = (\operatorname{rank}\Gamma)/2$. Unfortunately, condition (5.4) gives a small stability zone. For example, it necessarily breaks down for large values of the parameter γ, characterizing the intensity of the gyroscopic forces.

This research was supported financially by the Russian Foundation for Basic Research (99-01-01096) and the "Leading Scientific Schools" programme (00-15-96146).

REFERENCES

1. CHETAYEV, N. G., *The Stability of Motion. Papers on Analytical Mechanics*. Izd. Akad. Nauk SSSR, Moscow, 1962.
2. BULATOVIČ, R. M., The stability of linear potential gyroscopic systems in cases when the potential energy has a maximum. *Prikl. Mat. Mekh.* 1997, **61**, 3, 385–389.
3. HUSEYN, K., HAGEDORN, P. and TESCHER, W., On the stability of linear conservative gyroscopic systems. ZAMP 1983, **34**, 807–815.
4. BOLOTIN, S. V. and KOZLOV, V. V., Asymptotic solutions of the equations of dynamics. Vestnik MGU. Ser. 1, *Matematika, Mekhanika*, 1983, 6, 98–103.
5. POZHARITSKII, G. K., The unsteady motion of conservative holonomic systems. *Prikl. Mat. Mekh.*, 1956, **20**, 1, 429–433.
6. HILL, G. W. On the part of the motion of the lunar perigee which is a function of the mean motion of the Sun and Moon. *Acta Math.*, 1886, **VIII**, 1–36.
7. BOLOTIN, S. V., The Hill determinant of a periodic trajectory. *Vestnik MGU. Ser. 1, Matematika, Mekhanika*, 1988, 3, 30–34.
8. KOZLOV, V. V., The stabilization of the unstable equilibria of charges by intense magnetic fields. *Prikl. Mat. Mekh.* 1997, **61**, 3, 390–397.
9. YAKUBOVICH, V. A. and STARZHINSKII, V. M., *Linear Differential Equations with Periodic Coefficients and their Applications*. Nauka, Moscow, 1972.
10. ZEVIN, A. A., Some conditions for the existence and stability of periodic oscillations in non-linear non-autonomous Hamilton systems. *Prikl. Mat. Mekh.*, 1984, **48**, 4, 637–646.
11. KREIN, M. G., The criteria of stable boundedness of the solutions of periodic canonical systems. *Prikl. Mat. Mekh.*, 1955, **19**, 6, 641–680.

Translated by R.C.G.